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Structure of the confining solutions for SU(3)-Yang-Mills equations and confinement mechanism

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Abstract

Structure of exact solutions modelling confinement is discussed for SU(3)-Yang-Mills equations and uniqueness of such confining solutions is proved in a certain sense. Relationship of the obtained results to QCD and the confinement mechanism is considered. Incidentally the Wilson confinement criterion for the found solutions is verified and also numerical estimates for strength of magnetic colour field responsible for linear confinement are adduced in the ground state of charmonium.

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1 Introduction

As was remarked in Ref. [1], there exists a natural way of building meson spectroscopy and relativistic models of mesons which might be based on the exact solutions of the SU(3)-Yang-Mills equations modelling quark confinement, the so-called confining solutions, and also on the corresponding modulo square integrable solutions of the Dirac equation in those confining SU(3)-fields. The given approach is the direct consequence of the *relativistic* QCD (quantum chromodynamics) Lagrangian since the mentioned Yang-Mills and Dirac equations are derived just from the latter one. In Ref. [1] both the types of solutions were obtained and then in Ref. [2] they were successfully applied to the description of the charmonium spectrum. In its turn, the mentioned description pointed out the linear confinement to be (classically) governed by the magnetic colour field linear in r, the distance between quarks.

The results of Refs. [1,2] suggested the following mechanism of confinement to occur within the framework of QCD (at any rate, for mesons and quarkonia). The gluon exchange between quarks is realized in such a way that at large distances it leads to the confining SU(3)-field which may be considered classically (the gluon concentration becomes huge and gluons form the boson condensate – a classical field) and is a *nonperturbative* solution of the SU(3)-Yang-Mills equations. Under the circumstances mesons are the *relativistic bound states* described by the corresponding wave functions – *nonperturbative* modulo square integrable solutions of the Dirac equation in this confining SU(3)-field. For each meson there exists its own set of real constants (for more details see Section 2) a_j, A_j, b_j, B_j parametrizing the mentioned confining gluon field (the gluon condensate) and the corresponding wave functions while the latter ones also depend on μ_0, the reduced mass of the current masses of quarks forming meson. It is clear that constants $a_j, A_j, b_j, B_j, \mu_0$ should be extracted from experimental data. Further application of the approach to quarkonia (charmonium and bottomonium) in Refs. [3,4] confirmed the mentioned picture of linear confinement and, in particular, gave possibility to estimate the above gluon concentrations.

The aim of the present Letter is to specify a number of features in the above confinement scenario, in particular, to show that the confining solutions found in Ref. [1] and used in Refs. [2–4] are in essence the unique ones.

Further we shall deal with the metric of the flat Minkowski spacetime M that we write down [using the ordinary set of local spherical (r, ϑ, φ) or rectangular (Cartesian) (x, y, z) coordinates for spatial part in the forms

$$ds^2 = g_{\mu\nu} dx^\mu \otimes dx^\nu \equiv dt^2 - dr^2 - r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2) \,, \tag{1}$$

or

$$ds^2 = g_{\mu\nu} dx^\mu \otimes dx^\nu \equiv dt^2 - dx^2 - dy^2 - dz^2 \,, \tag{2}$$

so the components $g_{\mu\nu}$ take different values depending on the choice of coordinates. Besides we have $\delta = |\det(g_{\mu\nu})| = (r^2 \sin\vartheta)^2$ in spherical coordinates and the exterior differential $d = \partial_t dt + \partial_x dx + \partial_y dy + \partial_z dz$ or $d = \partial_t dt + \partial_r dr + \partial_\vartheta d\vartheta + \partial_\varphi d\varphi$ in the corresponding coordinates and we denote 3-dimensional vectors by bold font.

Throughout the paper we employ the system of units with $\hbar = c = 1$, unless explicitly stated otherwise.

2 Uniqueness of the confining solutions

If $A = A_\mu dx^\mu = A^a_\mu \lambda_a dx^\mu$ is a SU(3)-connection in the (trivial) three-dimensional bundle ξ over the Minkowski spacetime, where λ_c are the known Gell-Mann matrices, then we are interested in the confining solutions A of the SU(3)-Yang-Mills equations

$$d * F = g(*F \wedge A - A \wedge *F) , \tag{3}$$

while the curvature matrix (field strentgh) for the ξ-bundle is $F = dA + gA \wedge A = F^a_{\mu\nu} \lambda_a dx^\mu \wedge dx^\nu$ and $*$ means the Hodge star operator conforming to metric (1), g is a gauge coupling constant.

The confining solutions were defined in Ref. [1] as the spherically symmetric solutions of the Yang-Mills equations (3) containing only the components of the SU(3)-field which are Coulomb-like or linear in r. Additionally we shall impose the Lorentz condition on the sought solutions. The latter condition is necessary for quantizing the gauge fields consistently within the framework of perturbation theory (see, e. g. Ref. [5]), so we should impose the given condition that can be written in the form $\mathrm{div}(A) = 0$, where the divergence of the Lie algebra valued 1-form $A = A_\mu dx^\mu = A^a_\mu \lambda_a dx^\mu$ is defined by the relation (see, e. g. Refs. [6])

$$\mathrm{div}(A) = \frac{1}{\sqrt{\delta}} \partial_\mu (\sqrt{\delta} g^{\mu\nu} A_\nu) . \tag{4}$$

It should be noted the following. If writing down the Yang-Mills equations (3) in components then we shall be drowned in a sea of indices which will strongly hamper searching for one or another ansatz and make it to be practically immense. Using the Hodge star operator as well as the rules of external calculus makes the problem to be quite foreseeable and quickly leads to the aim. Let us remind, therefore, the properties of Hodge star operator we shall need (for more details see Refs. [6]).

2.1 Hodge star operator $*$ on Minkowski spacetime in spherical coordinates

Let M is a smooth manifold of dimension n so we denote an algebra of smooth functions on M as $F(M)$. In a standard way the spaces of smooth differential p-forms $\Lambda^p(M)$ $(0 \le p \le n)$ are defined over M as modules over $F(M)$. If a (pseudo)riemannian metric $G = ds^2 = g_{\mu\nu} dx^\mu \otimes dx^\nu$ is given on M in local coordinates $x = (x^\mu)$ then G can naturally be continued on spaces $\Lambda^p(M)$ by relation

$$G(\alpha, \beta) = \det\{G(\alpha_i, \beta_j)\} \tag{5}$$

for $\alpha=\alpha_1\wedge\alpha_2...\wedge\alpha_p$, $\beta=\beta_1\wedge\beta_2...\wedge\beta_p$, where for 1-forms $\alpha_i=\alpha^{(i)}_\mu dx^\mu$, $\beta_j=\beta^{(j)}_\nu dx^\nu$ we have $G(\alpha_i,\beta_j)=g^{\mu\nu}\alpha^{(i)}_\mu\beta^{(j)}_\nu$ with the Cartan's wedge (external) product $\wedge$. Under the circumstances the Hodge star operator $*$: $\Lambda^p(M)\to\Lambda^{n-p}(M)$ is defined for any $\alpha\in\Lambda^p(M)$ by

$$\alpha\wedge(*\alpha)=G(\alpha,\alpha)\omega_g \tag{6}$$

with the volume n-form $\omega_g=\sqrt{|\det(g_{\mu\nu})|}dx^1\wedge...dx^n$. For example, for 2-forms $F=F_{\mu\nu}dx^\mu\wedge dx^\nu$ we have

$$F\wedge *F=(g^{\mu\alpha}g^{\nu\beta}-g^{\mu\beta}g^{\nu\alpha})F_{\mu\nu}F_{\alpha\beta}\sqrt{\delta}\,dx^1\wedge dx^2\cdots\wedge dx^n,\ \mu<\nu,\ \alpha<\beta \tag{7}$$

with $\delta=|\det(g_{\mu\nu})|$. If s is the number of (-1) in a canonical presentation of quadratic form G then two most important properties of $*$ for us are

$$*^2=(-1)^{p(n-p)+s}\,, \tag{8}$$

$$*(f_1\alpha_1+f_2\alpha_2)=f_1(*\alpha_1)+f_2(*\alpha_2) \tag{9}$$

for any $f_1,f_2\in F(M)$, $\alpha_1,\alpha_2\in\Lambda^p(M)$, i. e., $*$ is a $F(M)$-linear operator. Due to (9) for description of $*$-action in local coordinates it is enough to specify $*$-action on the basis elements of $\Lambda^p(M)$, i. e. on the forms $dx^{i_1}\wedge dx^{i_2}\wedge...\wedge dx^{i_p}$ with $i_1<i_2<...<i_p$ whose number is equal to $C^p_n=\frac{n!}{(n-p)!p!}$.

The most important case of M in the given paper is the Minkowski spacetime with local coordinates t,r,ϑ,φ, where r,ϑ,φ stand for spherical coordinates on spatial part of M. The metric is given by (1) and we shall obtain the $*$-action on the basis differential forms according to (6)

$$*dt=r^2\sin\vartheta dr\wedge d\vartheta\wedge d\varphi,\ *dr=r^2\sin\vartheta dt\wedge d\vartheta\wedge d\varphi,$$

$$\begin{gathered}
*d\vartheta=-r\sin\vartheta dt\wedge dr\wedge d\varphi,\ *d\varphi=rdt\wedge dr\wedge d\vartheta,\\
(dt\wedge dr)=-r^2\sin\vartheta d\vartheta\wedge d\varphi\,,(dt\wedge d\vartheta)=\sin\vartheta dr\wedge d\varphi\,,\\
(dt\wedge d\varphi)=-\frac{1}{\sin\vartheta}dr\wedge d\vartheta\,,(dr\wedge d\vartheta)=\sin\vartheta dt\wedge d\varphi\,,\\
(dr\wedge d\varphi)=-\frac{1}{\sin\vartheta}dt\wedge d\vartheta\,,(d\vartheta\wedge d\varphi)=\frac{1}{r^2\sin\vartheta}dt\wedge dr\,,\\
(dt\wedge dr\wedge d\vartheta)=\frac{1}{r}d\varphi\,,(dt\wedge dr\wedge d\varphi)=-\frac{1}{r\sin\vartheta}d\vartheta,\\
*(dt\wedge d\vartheta\wedge d\varphi)=\frac{1}{r^2\sin\vartheta}dr,\ *(dr\wedge d\vartheta\wedge d\varphi)=\frac{1}{r^2\sin\vartheta}dt,
\end{gathered} \tag{10}$$

so that on 2-forms $*^2=-1$, as should be in accordance with (8).

At last it should be noted that all the above is easily over linearity continued on the matrix-valued differential forms (see, e. g., Ref. [7]), i. e., on the arbitrary linear combinations of forms $a_{i_1i_2...i_p}dx^{i_1}\wedge dx^{i_2}\wedge...\wedge dx^{i_p}$, where coefficients

$a_{i_1 i_2 \ldots i_p}$ belong to some space of matrices V, for example, a SU(3)-Lie algebra. But now the Cartan's wedge (external) product $\wedge$ should be understood as product of matrices with elements consisting of usual (scalar) differential forms. In the SU(3)-case, if T_a are matrices of generators of the SU(3)-Lie algebra in 3-dimensional representation, we continue the above scalar product G on the SU(3)-Lie algebra valued 1-forms $A = A^a_\mu T_a dx^\mu$ and $B = B^b_\nu T_b dx^\nu$ by the relation

$$G(A, B) = g^{\mu\nu} A^a_\mu B^b_\nu \mathrm{Tr}(T_a T_b)\,, \tag{11}$$

where Tr signifies the trace of a matrix, and, on linearity with the help of (5), G can be continued over any SU(3)-Lie algebra valued forms.

2.2 The confining solutions

Let us for definiteness put $T_a = \lambda_a$. Under this situation we can take the general ansatz of form

$$A = r^\mu \mathit{\Gamma} dt + A_r dr + A_\vartheta d\vartheta + r^\nu \mathit{\Delta} d\varphi\,, \tag{12}$$

where $A_\vartheta = r^\rho T$ and matrices $\mathit{\Gamma} = \alpha^a \lambda_a$, $\mathit{\Delta} = \beta^a \lambda_a$, $T = \gamma^a \lambda_a$, $A_r = f^a(r)\lambda_a$ with arbitrary real constants $\alpha^a, \beta^a, \gamma^a$ and arbitrary real functions $f^a(r)$. It could seem that there is a more general ansatz in the form $A = r^{\mu_a}\alpha^a\lambda_a dt + f^a(r)\lambda_a dr + r^{\rho_a}\gamma^a\lambda_a d\vartheta + r^{\nu_a}\beta^a\lambda_a d\varphi$ but somewhat more complicated considerations than the ones below show that all the same we should have $\mu_a = \mu, \nu_a = \nu, \rho_a = \rho$ for any a so we at once consider this condition to be fulfilled to avoid unnecessary complications. Then for the above ansatz (12) the Lorentz condition $\mathrm{div}(A) = 0$ takes the form

$$\partial_r(r^2 \sin\vartheta g^{rr} A_r) + \partial_\vartheta(r^2 \sin\vartheta g^{\vartheta\vartheta} A_\vartheta) = 0$$

which can be rewritten as

$$\partial_\vartheta(\sin\vartheta A_\vartheta) + \sin\vartheta \partial_r(r^2 A_r) = 0\,, \tag{13}$$

wherefrom it follows $\cot\vartheta r^\rho T + \partial_r(r^2 A_r) = 0$ while the latter entails

$$A_r = \frac{C}{r^2} - \frac{\cot\vartheta r^{\rho-1} T}{\rho + 1} \tag{14}$$

with a constant matrix C. Then we can see that it should put $C = T = 0$ or else A_r will not be spherically symmetric and the confining one where only the powers of r equal to ± 1 are admissible. As a result we come to the conclusion that one should put $A_r = A_\vartheta = 0$ in (12). After this we have ($[,]$ signifies matrix commutator) $F = dA + gA \wedge A = -\mu r^{\mu-1} \mathit{\Gamma} dt \wedge dr + \nu r^{\nu-1} \mathit{\Delta} dr \wedge d\varphi +$

$gr^{\mu+\nu}[\Gamma, \Delta]dt \wedge d\varphi$ which entails [with the help of (10)]

$$*F = \mu r^{\mu+1} \sin\vartheta \Gamma d\vartheta \wedge d\varphi - \frac{\nu r^{\nu-1}\Delta}{\sin\vartheta} dt \wedge d\vartheta - \frac{gr^{\mu+\nu}}{\sin\vartheta}[\Gamma, \Delta] dr \wedge d\vartheta$$

and the Yang-Mills equations (3) turn into

$$\mu(\mu+1)r^{\mu} \sin^2\vartheta \Gamma = g^2 r^{\mu+2\nu}[\Delta, [\Gamma, \Delta]] \,,$$

$$\nu(\nu-1)r^{\nu-2}\Delta = g^2 r^{2\mu+\nu}[\Gamma, [\Gamma, \Delta]] \,. \tag{15}$$

It is now not complicated to enumerate possibilities for obtaining the confining solutions in accordance with (15), where we should put $\mu = -1$, $\nu = 1$.

(1) Γ=0 or Δ=0. This situation does obviously not correspond to a confining solution
(2) $\Gamma = C_0\Delta$ with some constant C_0. This case conforms to that all the parameters α^a describing electric colour Coulomb field (see Subsection 2.3) and the ones β^a for linear magnetic colour field are proportional – the situation is not quite clear from physical point of view
(3) Matrices Γ, Δ are not equal to zero simultaneously and both matrices belong to Cartan subalgebra of SU(3)-Lie algebra. The parameters α^a, β^a of electric and magnetic colour fields are not connected and arbitrary, i. e. they should be chosen from experimental data. The given situation is the most adequate to the physics in question and the corresponding confining solution is in essence the same which has been obtained as far back as in Ref. [1] from other considerations.

One can slightly generalize the starting ansatz (12) taking it in the form $A = (r^{\mu}\Gamma + A')dt + (r^{\nu}\Delta + B')d\varphi$ with matrices $A' = A^a\lambda_a$, $B' = B^a\lambda_a$ and constants A^a, B^a. Then considerations along the same above lines draw the conclusion that the nontrivial confining solution is described by Γ, Δ, A', B' belonging to Cartan subalgebra.

Let us remind that, by definition, a Cartan subalgebra is a maximal abelian subalgebra in the corresponding Lie algebra, i. e., the commutator for any two matrices of the Cartan subalgebra is equal to zero (see, e.g., Ref. [8]). For SU(3)-Lie algebra the conforming Cartan subalgebra is generated by the Gell-Mann matrices λ_3, λ_8 which are

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{16}$$

Then it is easy to gain the only nontrivial solution in question in the form (which reflects the fact that for any matrix $\mathcal{T}$ from SU(3)-Lie algebra we have

$\mathrm{Tr}\,\mathcal{T}=0$)

$$A^3_t+\frac{1}{\sqrt{3}}A^8_t=-\frac{a_1}{r}+A_1,\,-A^3_t+\frac{1}{\sqrt{3}}A^8_t=-\frac{a_2}{r}+A_2,\,-\frac{2}{\sqrt{3}}A^8_t=\frac{a_1+a_2}{r}-(A_1+A_2),$$

$$A^3_\varphi+\frac{1}{\sqrt{3}}A^8_\varphi=b_1r+B_1,\,-A^3_\varphi+\frac{1}{\sqrt{3}}A^8_\varphi=b_2r+B_2,\,-\frac{2}{\sqrt{3}}A^8_\varphi=-(b_1+b_2)r-(B_1+B_2),\tag{17}$$

where real constants a_j, A_j, b_j, B_j parametrize the solution, and we wrote down the solution in the combinations that are just needed to insert into the corresponding Dirac equation (see Section 3). From here it follows one more form

$$A^3_t=[(a_2-a_1)/r+A_1-A_2]/2,\ A^8_t=[A_1+A_2-(a_1+a_2)/r]\sqrt{3}/2\,,$$
$$A^3_\varphi=[(b_1-b_2)r+B_1-B_2]/2, A^8_\varphi=[(b_1+b_2)r+B_1+B_2]\sqrt{3}/2\tag{18}$$

Clearly, the obtained results may be extended over all SU(N)-groups with $N\ge 2$ and even over all semisimple compact Lie groups since for them the corresponding Lie algebras possess just the only Cartan subalgebra. Also we can talk about the compact non-semisimple groups, for example, U(N). In the latter case additionally to Cartan subalgebra we have centrum consisting from the matrices of the form αI_N (I_N is the unit matrix $N\times N$) with arbitrary constant α. The most relevant physical cases are of course U(1)- and SU(3)-ones (QED and QCD), therefore we shall not consider further generalizations of the results obtained but let us write out the corresponding solution for U(1)-case which will useful for to interpret above solutions (17)–(18) in the more habitual physical terms. It should also be noted that the nontrivial confining solutions obtained exist at any gauge coupling constant g, i. e. they are essentially *nonperturbative* ones.

2.3 U(1)-case

Under this situation the Yang-Mills equations (3) turn into the second pair of Maxwell equations

$$d*F=0\tag{19}$$

with $F=dA$, $A=A_\mu dx^\mu$. We search for the solution of (19) in the form $A=A_t(r)dt+A_\varphi(r)d\varphi$. It is then easy to check that $F=dA=-\partial_rA_tdt\wedge dr+\partial_rA_\varphi dr\wedge d\varphi$ and according to (10) we get $*F=r^2\sin\vartheta\partial_rA_td\vartheta\wedge d\varphi-\frac{1}{\sin\vartheta}\partial_rA_\varphi dt\wedge d\vartheta$. From here it follows that (19) yields

$$\partial_r(r^2\partial_rA_t)=0,\ \partial^2_rA_\varphi=0\,,\tag{20}$$

and we write down the solutions of (20) as

$$A_t=\frac{a}{r}+A\,, A_\varphi=br+B\tag{21}$$

with some constants a, b, A, B parametrizing solutions (further for the sake of simplicity let us put $a = 1, b = 1$ GeV, $A = B = 0$).

To interpret solutions (21) in the more habitual physical terms let us pass on to Cartesian coordinates employing the relations

$$\varphi = \arctan(y/x),\ d\varphi = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy \tag{22}$$

which entails

$$A_\varphi d\varphi = -\frac{ry}{x^2+y^2}dx + \frac{rx}{x^2+y^2}dy \tag{23}$$

and we conclude that the solutions of (21) describe the combination of the electric Coulomb field with potential $\Phi = A_t$ and the constant magnetic field with vector-potential

$$\mathbf{A} = (A_x, A_y, A_z) = (-\frac{ry}{x^2+y^2}, \frac{rx}{x^2+y^2}, 0) = (-\frac{\sin\varphi}{\sin\vartheta}, \frac{\cos\varphi}{\sin\vartheta}, 0)\,, \tag{24}$$

which is *linear* in r in spherical coordinates and the 3-dimensional divergence $\mathrm{div}\mathbf{A} = 0$, as can be checked directly. Then Eqs. (19) in Cartesian coordinates takes the form

$$\Delta\Phi = 0,\ \mathrm{rotrot}\mathbf{A} = \Delta\mathbf{A} = 0 \tag{25}$$

with the Laplace operator $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$. At last, it is easy to check that the solution under consideration satisfies the Lorentz condition $\mathrm{div}(A) = 0$.

Practically the same considerations as the above ones in electrodynamics show that the solutions (17)–(18) describe the configuration of the electric Coulomb-like colour field (components A_t) with potentials Φ^3, Φ^8 and the constant magnetic colour field (components A_φ) with vector-potentials $\mathbf{A}^3, \mathbf{A}^8$ which are *linear* in r in spherical coordinates with 3-dimensional divergences $\mathrm{div}\mathbf{A}^{\mathbf{3,8}} = 0$ and $\Delta\Phi^{3,8} = 0$, $\mathrm{rotrot}\mathbf{A}^{\mathbf{3,8}} = \Delta\mathbf{A}^{\mathbf{3,8}} = 0$.

3 Relationship with QCD

The previous section leads us to the following problem: how to describe possible relativistic bound states in the obtained confining SU(3)-Yang-Mills fields? The sought description should be obviously based on the QCD-Lagrangian. Let us write down this Lagrangian (for one flavour) in arbitrary curvilinear (local) coordinates in Minkowski spacetime

$$\mathcal{L} = \overline{\Psi}\mathcal{D}\Psi - \mu_0\overline{\Psi}\Psi - \frac{1}{4}(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha})F^a_{\mu\nu}F^a_{\alpha\beta},\ \mu < \nu,\ \alpha < \beta \tag{26}$$

where, if denoting $S(M)$ and ξ, respectively, the standard spinor bundle and 3-dimensional vector one (equipped with a SU(3)-connection with the corresponding connection and curvature matrices $A = A_\mu dx^\mu = A^a_\mu \lambda_a dx^\mu$, $F = dA + gA \wedge A = F^a_{\mu\nu}\lambda_a dx^\mu \wedge dx^\nu$) over Minkowski spacetime, we can construct tensorial product $\Xi = S(M) \otimes \xi$. It is clear that Ψ is just a section of the latter bundle, i. e. Ψ can be chosen in the form $\Psi = (\Psi_1, \Psi_2, \Psi_3)$ with the four-dimensional Dirac spinors Ψ_j representing the j-th colour component while $\overline{\Psi} = \Psi^\dagger(\gamma^0 \otimes I_3)$ is the adjont spinor,($\dagger$) stands for hermitian conjugation, $\otimes$ means tensorial product of matrices, μ_0 is a mass parameter, $\mathcal{D}$ is the Dirac operator with coefficients in ξ (see below). At last, we have the condition $\mathrm{Tr}(\lambda_a\lambda_b) = 2\delta_{ab}$ so that the third addendum in (26) has the form $G(F, F)/8$ with G of (11), where coefficient 1/8 is chosen from physical considerations.

From general considerations (see, e. g., Ref. [9]) the explicit form of the operator $\mathcal{D}$ in local coordinates x^μ on Minkowski spacetime can be written as follows

$$\mathcal{D} = i(\gamma^e \otimes I_3)E^\mu_e \left(\partial_\mu \otimes I_3 - \frac{1}{2}\omega_{\mu ab}\gamma^a\gamma^b \otimes I_3 - igA_\mu\right),\ a < b, \tag{27}$$

where the forms $\omega_{ab} = \omega_{\mu ab}dx^\mu$ obey the Cartan structure equations $de^a = \omega^a{}_b \wedge e^b$, while the orthonormal basis $e^a = e^a_\mu dx^\mu$ in cotangent bundle and dual basis $E_a = E^\mu_a \partial_\mu$ in tangent bundle are connected by the relations $e^a(E_b) = \delta^a_b$. At last, matrices γ^a represent the Clifford algebra of the quadratic form $Q_{1,3} = x_0^2 - x_1^2 - x_2^2 - x_3^2$ in $\mathbb{C}^4$. It should be noted that Greek indices $\mu, \nu, ...$ are raised and lowered with $g_{\mu\nu}$ of (1) or its inverse $g^{\mu\nu}$ and Latin indices $a, b, ...$ are raised and lowered by $\eta_{ab} = \eta^{ab}$= diag(1,-1,-1,-1) except for Latin indices connected with SU(3)-Lie algebras, so that $e^a_\mu e^b_\nu g^{\mu\nu} = \eta^{ab}$, $E^\mu_a E^\nu_b g_{\mu\nu} = \eta_{ab}$ and so on but $\lambda_a = \lambda^a$.

Under the circumstances we can obtain the following equations according to the standard prescription of Lagrange approach from (26)

$$\mathcal{D}\Psi = \mu_0\Psi\ , \tag{28}$$

$$d * F = g(*F \wedge A - A \wedge *F) + gJ\ , \tag{29}$$

where the source J (a nonabelian SU(3)-current) is

$$J = j^a_\mu \lambda_a * (dx^\mu) = *j = *(j^a_\mu \lambda_a dx^\mu) = *(j^a \lambda_a) \tag{30}$$

with currents

$$j^a = j^a_\mu dx^\mu = \overline{\Psi}(\gamma_\mu \otimes I_3)\lambda^a \Psi\, dx^\mu\ ,$$

so summing over $a = 1, ..., 8$ is implied in (26) and (30).

When using the relation (see, e. g. Refs. [10]) $\gamma^c E^\mu_e \omega_{\mu ab}\gamma^a\gamma^b = \omega_{\mu ab}\gamma^\mu\gamma^a\gamma^b = -\mathrm{div}(\gamma)$ with matrix 1-form $\gamma = \gamma_\mu dx^\mu$ [where div is defined by relation (4)]

and also the fact that $(\gamma^\mu)^\dagger\gamma^0=\gamma^0\gamma^\mu$, the Dirac equation for spinor $\overline{\Psi}$ will be

$$i\partial_\mu\overline{\Psi}(\gamma^\mu\otimes I_3)+\frac{i}{2}\overline{\Psi}\mathrm{div}(\gamma)\otimes I_3-g\overline{\Psi}(\gamma^\mu\otimes I_3)A^a_\mu\lambda_a=-\mu_0\overline{\Psi}\,. \tag{28$'$}$$

Then multiplying (28) by $\overline{\Psi}\lambda_a$ from left and (28′) by $\lambda_a\Psi$ from right and adding the obtained equations, we get $\mathrm{div}(j^a)=\mathrm{div}(j)=0$ if spinor Ψ obeys the Dirac equation (28).

The question now is how to connect the sought relativistic bound states with the system (28)–(29). To understand it let us apply to the experience related with QED. In the latter case Lagrangian looks like (26) with changing group SU(3)→U(1) so Ψ will be just a four-dimensional Dirac spinor. Then, as is known (see, e. g. Ref. [11]), when passing on to the nonrelativistic limit the Dirac equation (28) converts into the Pauli equation and further, if neglecting the particle spin, into the Schrödinger equation, parameter μ_0 becoming the reduced mass of two-body system. The modulo square integrable solutions of the Schrödinger equation just describe bound states of a particle with mass μ_0, or, that is equivalent, of the corresponding two-body system. Historically, however, everything was just vice versa. At first there appeared the Schrödinger equation, then the Pauli and Dirac ones and only then the QED Lagrangian. In its turn, possibility of writing two-body Schrödinger equation on the whole owed to the fact that the corresponding two-body problem in classical nonrelativistic (newtonian) mechanics was well posed and actually quantizing the latter gave two-body Schrödinger equation. Another matter was Dirac equation. Up to now nobody can say what two-body problem in classical relativistic (einsteinian) mechanics could correspond to Dirac equation. The fact is that the two-body problem in classical relativistic mechanics has so far no single-valued statement. Conventionally, therefore, Dirac equation in QED is treated as the relativistic wave equation describing one particle with spin one half in an external electromagnetic field.

There is, however, one important exclusion – the hydrogen atom. When solving the Dirac equation here one considers mass parameter μ_0 to be equal to the electron mass and one gets the so-called Sommerfeld formula for hydrogen atom levels which passes on to the standard Schrödinger formula for hydrogen atom spectrum in nonrelativistic limit (for more details see, e. g. Ref. [11]). But in the Schrödinger formula mass parameter μ_0 is equal to the reduced mass of electron and proton. As a consequence, it is tacitly supposed that in Dirac equation the mass parameter should be equal to the same reduced mass of electron and proton as in Schrödinger equation. Just the mentioned reduced mass is approximately equal to that of electron but, exactly speaking, it is not the case. We remind that for the problem under discussion (hydrogen atom) the external field is the Coulomb electric one between electron and proton, essentially nonrelativistic object in the sense that it does not vanish in nonrelativistic limit at $c\to\infty$. If now to place hydrogen atom in a magnetic field

then obviously spectrum of bound states will also depend on parameters describing the magnetic field. The latter, however, is essentially relativistic object and vanishes at $c \to \infty$ because, as is well known, in the world with $c = \infty$ there exist no magnetic fields (see any elementary textbook on physics, e. g. Ref. [12]). But it is clear that spectrum should as before depend on μ_0 as well and we can see that μ_0 is the same reduced mass as before since in nonrelativistic limit we again should come to the hydrogen atom spectrum with the reduced mass. So we can draw the conclusion that if an electromagnetic field is a combination of electric Coulomb field between two charged elementary particles and some magnetic field (which may be generated by the particles themselves) then there are certain grounds to consider the given (quantum) two-body problem to be equivalent to the one of motion for one particle with usual reduced mass in the mentioned electromagnetic field. As a result, we can use the Dirac equation for finding possible relativistic bound states for such a particle implying that this is really some description of the corresponding two-body problem.

Actually in QED the situaion is just as the described one but magnetic field is usually weak and one may restrict oneself to some corrections from this field to the nonrelativistic Coulomb spectrum (e. g., in the Seemann effect). If the magnetic field is strong then one should solve just Dirac equation in a nonperturbative way (see, e.g. Ref. [13]). The latter situation seems to be natural in QCD where the corresponding (colour) magnetic field should be very strong (see Section 4) because just it provides linear confinement of quarks as we could see above (see also Refs. [2–4]) and the given field also vanishes in nonrelativistic limit (for more details see Refs. [2,3]).

At last, we should make an important point that in QED the mentioned electromagnetic field is by definition always a solution of the Maxwell equations so within QCD we should require the confining SU(3)-field to be a solution of Yang-Mills equations. Consequently, returning to the system (28)–(29), we can suggest to decribe relativistic bound states of two quarks (mesons) in QCD by the compatible solutions of the given system. To be more precise, the meson wave functions should be the *nonperturbative* modulo square integrable solutions of Dirac equation (28) (with the above reduced mass μ_0) in the confining SU(3)-Yang-Mills field being a *nonperturbative* solution of (29). In general case, however, the analysis of (29) is difficult because of availability of the nonabelian current J of (30) in the right-hand side of (29) but we may use the circumstance that the corresponding modulo square integrable solutions of Dirac equation (28) might consist from the components of form $\Psi_j \sim r^{\alpha_j} e^{-\beta_j r}$ with some $\alpha_j > 0$, $\beta_j > 0$ which entails all the components of the current J to be modulo $<< 1$ at each point of Minkowski space. The latter will allow us to put $J \approx 0$ and we shall come to the problem of finding the confining solutions for the Yang-Mills equations of (29) with $J = 0$ whose unique nontrivial form has been described in previous Section and after inserting the

found solutions into Dirac equation (28) we should require the corresponding solutions of Dirac equation to have the above necessary behaviour. Under the circumstances the problem becomes self-consistent and can be analyzable and this has been done actually in Ref. [1].

It is clear that all the above considerations can be justified only by comparison with experimental data but now we obtain some intelligible programme of further activity which has been partly realized in Refs. [2–4].

4 Verification of confinement criterion and estimates of colour magnetic field strength

To illustrate some of the above let us verify the Wilson confinement criterion [14] for the confining solutions (17)–(18) and also adduce numerical estimates for strength of colour magnetic field responsible for linear confinement in the ground state of charmonium.

4.1 Verification of confinement criterion

The Wilson confinement criterion is in essence the assertion that the so-called Wilson loop $W(c)$ should be subject to the area law for the confining gluonic field configuration. In its turn, the latter law is equivalent to the fact that energy $E(R)$ of the mentioned configuration (gluon condensate) is linearly increasing with R, a characteristic size of some volume V containing the condensate. We can easily evaluate $E(R)$ for solutions (17)–(18) using the T_{00}-component (volumetric energy density) of the energy-momentum tensor for a SU(3)-Yang-Mills field

$$T_{\mu\nu} = \frac{1}{4\pi}\left(-F^a_{\mu\alpha}\,F^a_{\nu\beta}\,g^{\alpha\beta} + \frac{1}{4}F^a_{\beta\gamma}\,F^a_{\alpha\delta}g^{\alpha\beta}g^{\gamma\delta}g_{\mu\nu}\right)\,. \tag{31}$$

It is not complicated to obtain the curvature matrix (field strentgh) corresponding to the solution (17)–(18)

$$F = F^a_{\mu\nu}\lambda_a dx^\mu \wedge dx^\nu = -\partial_r(A^a_t\lambda_a)dt\wedge dr + \partial_r(A^a_\varphi\lambda_a)dr\wedge d\varphi\,, \tag{32}$$

which entails the only nonzero components

$$F^3_{tr} = \frac{a_1-a_2}{2r^2},\; F^8_{tr} = \frac{(a_1+a_2)\sqrt{3}}{2r^2},\; F^3_{r\varphi} = \frac{b_1-b_2}{2},\; F^8_{r\varphi} = \frac{(b_1+b_2)\sqrt{3}}{2} \tag{33}$$

and, in its turn,

$$T_{00}\equiv T_{tt}=\frac{1}{4\pi}\left\{\frac{3}{4}\left[(F^3_{tr})^2+(F^8_{tr})^2\right]+\frac{1}{4r^2\sin^2\vartheta}\left[(F^3_{r\varphi})^2+(F^8_{r\varphi})^2\right]\right\}=$$

$$\frac{3}{16\pi}\left(\frac{a_1^2+a_1a_2+a_2^2}{r^4}+\frac{b_1^2+b_1b_2+b_2^2}{3r^2\sin^2\vartheta}\right)\equiv\frac{\mathcal{A}}{r^4}+\frac{\mathcal{B}}{r^2\sin^2\vartheta}\qquad(34)$$

with $\mathcal{A}>0$, $\mathcal{B}>0$. Let V be the volume between two concentric spheres with radii $R_0<R$. Then $E(R)=\int_V T_{00}\sqrt{\delta}d^3x=\int_V T_{00}r^2\sin\vartheta dr d\vartheta d\varphi$. As was discussed in Ref.[4], the notion of classical gluonic field is applicable only at distances $>>\lambda_B$, the de Broglie wavelength of quark, so we may take R_0 to be of order of a characteristic size of hadron (meson). Besides it should be noted that classical T_{00} of (34) has a singularity along z-axis ($\vartheta=0,\pi$) and we have to introduce some angle ϑ_0 whose physical meaning is to be clarified a little below. Under the circumstances, with employing the relations $\int d\vartheta/\sin\vartheta=\ln\tan\vartheta/2$, $\tan\vartheta/2=\sin\vartheta/(1+\cos\vartheta)=(1-\cos\vartheta)/\sin\vartheta$, we shall have

$$E(R)=\int\limits_{R_0}^{R}\int\limits_{\vartheta_0}^{\pi-\vartheta_0}\int\limits_{0}^{2\pi}\left(\frac{\mathcal{A}}{r^2}+\frac{\mathcal{B}}{\sin^2\vartheta}\right)\sin\vartheta dr d\vartheta d\varphi=E_0-\frac{4\pi\mathcal{A}}{R}+2\pi\mathcal{B}R\ln\frac{1+\cos\vartheta_0}{1-\cos\vartheta_0}\qquad(35)$$

with $E_0=\frac{4\pi\mathcal{A}}{R_0}-2\pi\mathcal{B}R_0\ln\frac{1+\cos\vartheta_0}{1-\cos\vartheta_0}$. It is clear that at $R\to\infty$ we get $E(R)\sim R$, i. e. the area law is fulfilled.

4.2 Estimates of colour magnetic field strength

To estimate R_0 and ϑ_0 under real physical situation let us consider the ground state of charmonium $\eta_c(1S)$ and apply the ideology of Section 3 for describing the given state. We use the parametrization of relativistic spectrum of charmonium from Ref. [4]. Namely, Table 1 contains necessary parameters of solutions (17)–(18) for the sought estimates. As for parameters $A_{1,2}$ of (17)–

Table 1
Gauge coupling constant, mass parameter μ_0 and parameters of the confining SU(3)-gluonic field for charmonium.

g	μ_0 (GeV)	a_1	a_2	b_1 (GeV)	b_2 (GeV)	B_1	B_2
0.46900	0.62500	2.21104	-0.751317	20.2395	-12.6317	6.89659	6.89659

(18) then we can put them equal to zero [4]. One can note that the reduced mass parameter μ_0 is consistent with the present-day experimental limits [15] where the current mass of c-quark ($2\mu_0$) is accepted between 1.1 GeV and 1.4 GeV. Then wave function $\Psi=(\Psi_1,\Psi_2,\Psi_3)$ of the state under consideration

[as a modulo square integrable solution of Dirac equation (28) in the field (17)–(18)] is given by the form (with Pauli matrix σ_1)

$$\Psi_j = e^{i\omega_j t} r^{-1} \begin{pmatrix} F_{j1}(r)\Phi_j(\vartheta,\varphi) \\ F_{j2}(r)\sigma_1\Phi_j(\vartheta,\varphi) \end{pmatrix}, j = 1,2,3 \tag{36}$$

with the 2D eigenspinor $\Phi_j = \begin{pmatrix} \Phi_{j1} \\ \Phi_{j2} \end{pmatrix}$ of the euclidean Dirac operator on the unit sphere $\mathbb{S}^2$. The explicit form of Φ_j is not needed here and can be found in Ref. [4]. The radial parts are

$$F_{j1} = C_j P_j r^{\alpha_j} e^{-\beta_j r}\left(1 - \frac{gb_j}{\beta_j}\right), F_{j2} = iC_j Q_j r^{\alpha_j} e^{-\beta_j r}\left(1 + \frac{gb_j}{\beta_j}\right) \tag{37}$$

with $\alpha_j = \sqrt{\Lambda_j^2 - g^2 a_j^2}$, $\Lambda_j = -1 - gB_j$, $\beta_j = \sqrt{\mu_0^2 - \omega_j^2 + g^2 b_j^2}$, $P_j = gb_j + \beta_j$, $Q_j = \mu_0 + \omega_j$ where $a_3 = -(a_1 + a_2), b_3 = -(b_1 + b_2), B_3 = -(B_1 + B_2)$ and constants C_j are determined from the normalization conditions $\int_0^\infty (|F_{j1}|^2 + |F_{j2}|^2)dr = 1/3$. Energy of the ground state is obtained as

$$\epsilon = \sum_{j=1}^{3} \omega_j \equiv \frac{-\Lambda_1 g^2 a_1 b_1 + \alpha_1|\Lambda_1|\mu_0}{\Lambda_1^2} + \frac{-\Lambda_2 g^2 a_2 b_2 + \alpha_2|\Lambda_2|\mu_0}{\Lambda_2^2} + \frac{-\Lambda_3 g^2 a_3 b_3 - \alpha_3|\Lambda_3|\mu_0}{\Lambda_3^2} = 2.9796\ \text{GeV}. \tag{38}$$

Then we can put $R_0 = \frac{1}{3}\sum_{j=1}^{3} 1/\beta_j \approx 0.0399766$ fm and, adding the rest energies of quarks $2(2\mu_0)$ to E_0 of (35), we consider $\epsilon = E_0 + 4\mu_0 = 2.9796$ GeV which entails $\vartheta_0 \approx 0.723388$ and one may consider ϑ_0 to be the parameter determining some cone $0 \le \vartheta \le \vartheta_0$ so the quark emits gluons outside of the cone. It is evident that the Wilson criterion is purely classic one: quantum mechanically the stay probability of quarks at distances $> R_0$ is of order $\sum_{j=1}^{3}(|F_{j1}|^2 + |F_{j2}|^2)$ and is damped exponentially fast at $R > R_0$. Just the colour magnetic field defines this damping through the coefficients β_j.

At last, using the Hodge star operator in 3-dimensional euclidean space [where $ds^2 = g_{\mu\nu}dx^\mu \otimes dx^\nu \equiv dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)$, $*(dr \wedge d\vartheta) = \sin\vartheta d\varphi$, $*(dr \wedge d\varphi) = -d\vartheta/\sin\vartheta$, $*(d\vartheta \wedge d\varphi) = dr/(r^2 \sin\vartheta]$, we can confront components $F^3_{r\varphi}, F^8_{r\varphi}$ of (33) with 3-dimensional 1-forms of the magnetic colour field

$$\mathbf{H}^3 = -\frac{b_1 - b_2}{2\sin\vartheta}d\vartheta\,, \mathbf{H}^8 = -\frac{(b_1 + b_2)\sqrt{3}}{2\sin\vartheta}d\vartheta$$

that are modulo equal to $\sqrt{g^{\mu\nu}\mathbf{H}^{3,8}_\mu \mathbf{H}^{3,8}_\nu}$ or

$$H^3 = \frac{|b_1 - b_2|}{2r\sin\vartheta},\ H^8 = \frac{|b_1 + b_2|\sqrt{3}}{2r\sin\vartheta}\,, \tag{39}$$

where we shall put $\sin\vartheta = 1$ for simplicity. Table 2 contains estimates of magnetic colour field strength in the ground state of charmonium whereas the

Bohr radius $a_0 = 0.529177249 \cdot 10^5$ fm [15]. Also when calculating we applied the relations 1 GeV$^{-1} \approx 0.21030893$ fm , 1 T $\approx 0.692508 \times 10^{-15}$ GeV^2.

Table 2
Magnetic colour field strengths in the ground state of charmonium.

$\eta_c(1S)$:	$R_0 = 0.0399766$ fm	
r (fm)	H^3 (T)	H^8 (T)
$0.1R_0$	$0.124857 \cdot 10^{19}$	$0.500516 \cdot 10^{18}$
R_0	$0.124857 \cdot 10^{18}$	$0.500516 \cdot 10^{17}$
$10R_0$	$0.124857 \cdot 10^{17}$	$0.500516 \cdot 10^{16}$
1.0	$0.499137 \cdot 10^{16}$	$0.200089 \cdot 10^{16}$
a_0	$0.943232 \cdot 10^{11}$	$0.378114 \cdot 10^{11}$

It is seen that at the characteristic scales of the charmonium ground state the strength of magnetic colour field responsible for linear confinement reaches huge values of order 10^{17}–10^{18} T. For comparison one should notice that the most strong magnetic fields known at present have been discovered in magnetic neutron stars, pulsars (see, e. g., Ref. [16]) where the corresponding strengths can be of order 10^9–10^{10} T. So the characterictic feature of confinement is really very strong magnetic colour field between quarks. In a certain sense the essence of confinement can be said to be just in enormous gluon concentrations and magnetic colour field strentghs in space around quarks.

5 Concluding remarks

The aim of this Letter we pursued was to specify a scenario of linear confinement of quarks, at any rate, for mesons and quarkonia. As we could see, crucial step consisted in studying exact solutions of the SU(3)-Yang-Mills equations modelling confinement. Namely structure of the confining solutions for SU(3)-Yang-Mills equations prompts to the idea that linear confinement should be governed by magnetic colour field. But the latter is essentially relativistic object vanishing in nonrelativistic limit which implies the relativistic effects to be extremely important for the confinement mechanism [2–4].

Further specification of the confinement mechanism can consist in the inquiry answer: what the way of gluon exchange between quarks is? The matter concerns such a modification of gluon propagator which might correspond to linear confinement at large distances. As was remarked in Ref. [2], the sought modification should be carried out on the basis of the exact solutions of the

SU(3)-Yang-Mills equations modelling quark confinement. We hope to consider this modification elsewhere.

There is also an interesting possibility of indirect experimental verification of the confinement mechanism under discussion. Really solutions (21) point out the confinement phase could be in electrodynamics as well. Though there exist no elementary charged particles generating a constant magnetic field linear in r, the distance from particle, after all, if it could generate this elecromagnetic field configuration in laboratory then one might study motion of macroscopic charged particles in that field. The confining properties of the mentioned field should be displayed at classical level too but the exact behaviour of particles in this field requires certain analysis of the corresponding classical equations of motion.

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